Fibre bundle formulation of relativistic quantum mechanics

II. Covariant approach

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Contents

Abstract

We propose a new fibre bundle formulation of the mathematical base of relativistic quantum mechanics. At the present stage the bundle form of the theory is equivalent to its conventional one, but it admits new types of generalizations in different directions.

In the present second part of our investigation, we consider the covariant approach to bundle description of relativistic quantum mechanics. In it the wavefunctions are replaced with (state) sections of a suitably chosen vector bundle over space-time whose (standard) fibre is the space of the wavefunctions. Now the quantum evolution is described as a linear transportation (by means of the evolution transport along the identity map of the space-time) of the state sections in the bundle (total) space. The equations of these transportations turn to be the bundle versions of the corresponding relativistic wave equations. Connections between the (retarded) Green functions of these equations and the evolution operators and transports are found. Especially the Dirac and Klein-Gordon equations are considered.

1. Introduction

This paper is a second part of our investigation devoted to the fibre bundle description of relativistic quantum mechanics. It is a straightforward continuation of [**?**].

The developed in [**?**] bundle formalism for relativistic quantum wave equations has the deficiency that it is not explicitly covariant; so it is not in harmony with the relativistic theory it represents. This is a consequence of the direct applications of the bundle methods developed for the nonrelativistic region, where they work well enough, to the relativistic one. The present paper is intended to mend this 'defect'. Here we develop an appropriate covariant bundle description of relativistic quantum mechanics which corresponds to the natural character of this theory. The difference between the time-dependent and covariant approaches is approximately the same as the one between the Hamiltonian and Lagrangian derivation (and forms) of some relativistic wave equation.

The organization of the material is the following.

Sect. 2 contains a detailed covariant application of the ideas of the bundle description of nonrelativistic quantum mechanics (see, e.g. [**?**, sect. 2] or [**?**, **?**]) to Dirac equation. The bundle, where Dirac particles 'live', is a vector bundle over space-time with the space of 4-spinors as a fibre; so here we again work with the 4-spinor bundle of [**?**, sect. 4], but now the evolution of a Dirac particle is described via the *(Dirac) evolution transport* which is a *linear transport along the identity map of space-time*. The state of a Dirac particle is represented by a *section (not along paths!)* of the 4-spinor bundle and is (linearly) transported by means of Dirac evolution transport in this bundle. The Dirac equation itself is transformed into a system of four equations uniquely describing this transportation. We write different equivalent (bundle or not) covariant versions of Dirac equation. In particular, we have found an interesting *algebraic* form of the Dirac equation connecting the coefficients of Dirac evolution transport and the Dirac analogue of the Hamiltonian corresponding to a covariant Schrödinger-like form of Dirac equation. This analogue plays a rôle of a gauge field and from it the whole theory can be recovered.

In Sect. 3 we apply the covariant bundle approach to Klein-Gordon equation. For this purpose we present a 5-dimensional representation of this equation as a first-order Dirac-like equation to which *mutatis mutandis* the theory of Sect. 2 can be transferred practically without changes.

The goal of Sect. 4 is to be revealed some connections between the retarded Green functions (≡propagators) of the relativistic wave equations and the corresponding to them evolution operators and transports. Generally speaking, the evolution operators (resp. transports) admit representation as integral operators, the kernel of which is connected in a simple manner with the retarded Green function (resp. Green morphism of a bundle).

Subsect. 4.1 contains a brief general consideration of the Green functions and their connection with the evolution transports, if any. In Subsect. 4.2, 4.3, and 4.4 we derive the relations mentions for Schrödinger, Dirac, and Klein-Gordon equations, respectively.

Sect. 5 closes the paper with a brief summary of the main ideas underlying the bundle description of relativistic quantum mechanics.

Appendix A contains some mathematical results concerning the theory of (linear) transports along maps required for the present investigation.

In Appendix B are given certain formulae concerning matrix operators, i.e. matrices whose elements are operators, which naturally arise in relativistic quantum mechanics.

The notation of the present work is the the same as the one in [**?**] and we are not going to recall it here.

The references to sections, equations, footnotes etc. from [**?**] are obtained from their sequential numbers in [**?**] by adding in front of them the Roman one (I) and a dot as a separator. For instance, Sect. I.4 and (I.5.2) mean respectively section 4 and equation (5.2) (equation 2 in Sect. 5) of [**?**].

Below, for reference purposes, we present a list of some essential equations of [**?**] which are used in this paper. Following the just given convention, we retain their original reference numbers.

$$\psi(t) = \mathcal{U}(t,t_0)\psi(t_0), \tag{I.2.2}$$

$$\Psi_\gamma\colon t \to \Psi_\gamma(t) = l^{-1}_{\gamma(t)}\big(\psi(t)\big), \qquad H_\gamma\colon t \to H_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{H} \circ l_{\gamma(t)}, \tag{I.2.3}$$

$$U_\gamma(t,s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}(t,s) \circ l_{\gamma(s)}\colon F_{\gamma(s)} \to F_{\gamma(t)}, \qquad s,t \in J, \tag{I.2.4}$$

$$\boldsymbol{\Gamma}_\gamma(t) := \big[\Gamma^b{}_a(t;\gamma)\big] = -\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^{\mathbf{m}}_\gamma(t), \tag{I.2.10}$$

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)}\colon F_{\gamma(t)} \to F_{\gamma(t)}. \tag{I.2.11}$$

2. Dirac equation

The covariant Dirac equation [**?**, sect. 2.1.2], [**?**, chapter XX, § 8] for a (spin $\frac{1}{2}$) particle with mass m and electric charge e in an external electromagnetic field with 4-potential $\mathcal{A}^\mu$ is

$$(\mathrm{i}\hbar \not{D} - mc\mathbb{1}_4)\psi = 0, \qquad \not{D} := \gamma^\mu D_\mu, \quad D_\mu := \partial_\mu - \frac{e}{\mathrm{i}\hbar c}\mathcal{A}_\mu. \tag{2.1}$$

Here $i \in \mathbb{C}$ is the imaginary unit, $\hbar$ is the Plank constant (divided by 2π), $\mathbb{1}_4 = \operatorname{diag}(1,1,1,1)$ is the 4×4 unit matrix, $\psi := (\psi^0,\psi^1,\psi^2,\psi^3)$ is (the matrix of the components of) a 4-spinor, γ^μ, $\mu = 0,1,2,3$, are the well known Dirac γ-matrices [**?**, **?**, **?**], and c is the velocity of light in vacuum. Since it is a first order partial differential equation, it admits *evolution operator* $\mathcal{U}$ connecting the values of its solutions at different spacetime points. More precisely, if $x_1, x_2 \in M$, M being the spacetime, i.e. the Minkowski space M^4, then

$$\psi(x_2) = \mathcal{U}(x_2, x_1)\psi(x_1). \tag{2.2}$$

Here $\mathcal{U}(x_2, x_1)$ is a 4×4 matrix operator (see Appendix B) defined as the unique solution of the initial-value problem[1]

$$(\mathrm{i}\hbar \not{D}_x - mc\mathbb{1}_4)\mathcal{U}(x, x_0) = 0, \qquad \mathcal{U}(x_0, x_0) = \mathsf{id}_{\mathcal{F}}, \quad x, x_0 \in M \tag{2.3}$$

where $\not{D} = \not{\partial} - \frac{e}{\mathrm{i}\hbar c}\not{\mathcal{A}}$ with $\not{\partial} := \gamma^\mu \partial/\partial x^\mu$ and $\not{\mathcal{A}} := \gamma^\mu \mathcal{A}_\mu$, $\mathcal{F}$ is the space of 4-spinors, and id_X is the identity map of a set X.

Suppose (F, π, M) is a vector bundle with (total) bundle space F, projection $\pi\colon F \to M$, fibre $\mathcal{F}$, and isomorphic fibres $F_x := \pi^{-1}(x)$, $x \in M$. There exist linear isomorphisms $l_x : F_x \to \mathcal{F}$ which we assume to be diffeomorphisms; so $F_x = l_x^{-1}(\mathcal{F})$ are 4-dimensional vector spaces.

To a state vector (spinor) ψ we assign a C^1 section[2] Ψ of (F, π, M), i.e. $\Psi \in \mathrm{Sec}^1(F, \pi, M)$, by (cf. (I.2.3))

$$\Psi(x) := l_x^{-1}\big(\psi(x)\big) \in F_x := \pi^{-1}(x), \quad x \in M. \tag{2.4}$$

Since in (F, π, M) the state of a Dirac particle is described by Ψ, we call it *state section*; resp. (F, π, M) is the *4-spinor bundle*. The description of Dirac particle via Ψ will be called *bundle description*. If it is known, the conventional spinor description is achieved by the spinor

$$\psi(x) := l_x\big(\Psi(x)\big) \in \mathcal{F}. \tag{2.5}$$

Now the analogue of (2.2) is

$$\Psi(x_2) = U(x_2, x_1)\Psi(x_1), \qquad x_2, x_1 \in M. \tag{2.6}$$

with (cf. (I.2.4))

$$U(y, x) = l_y^{-1} \circ \mathcal{U}(y, x) \circ l_x \colon F_x \to F_y, \qquad x, y \in M. \tag{2.7}$$

Since $\mathcal{U}$ is a 4×4 matrix operator satisfying (2.3) and, as can easily be proved, the equality

$$\mathcal{U}(x_3, x_1) = \mathcal{U}(x_3, x_2) \circ \mathcal{U}(x_2, x_1), \qquad x_1, x_2, x_3 \in M, \tag{2.8}$$

[1]Generally $\mathcal{U}$ is an integral matrix operator whose kernel for the free Dirac equation is given in [**?**, sect.2.5.1, equations (2.107) and (2.108)].

[2]In contrast to the time-dependent approach [**?**] and nonrelativistic case [**?**] now Ψ is simply a section, not section along paths [**?**]. Physically this corresponds to the fact that quantum objects do not have world lines (trajectories) in a classical sense [**?**].

is valid, the map (2.7) is linear and:

$$U(x_3, x_1) = U(x_3, x_2) \circ U(x_2, x_1), \qquad x_1, x_2, x_3 \in M, \tag{2.9}$$

$$U(x, x) = \mathsf{id}_{F_x}, \qquad x \in M. \tag{2.10}$$

Consequently, by definition A.1, the map $U\colon (y, x) \to U(y, x) = K^{\mathsf{id}_M}_{x\to y}$ is a linear transport along the identity map id_M of M in the bundle (F, π, M). Alternatively, as it is mentioned in Appendix A, this means that $L^\gamma\colon (t, s) \to L^\gamma_{s\to t} := U(\gamma(t), \gamma(s))$, $s, t \in J$ is a flat linear transport along $\gamma\colon J \to M$ in (F, π, M). We call U the *(Dirac) evolution transport*.

Equation (2.6) simply means that Ψ is *U-transported (along id_M) section* of (F, π, M) (cf. [**?**, definition 5.1]). Writing (A.10) for the evolution transport U and applying the result to a state section given by (2.4), we easily can prove that (2.6) is equivalent to

$$\mathcal{D}_\mu \Psi = 0, \qquad \mu = 0, 1, 2, 3 \tag{2.11}$$

where, for brevity, we have put $\mathcal{D}_\mu := \mathcal{D}^{\mathsf{id}_M}_{x^\mu}$ which is the μ-th partial (section-)derivation along the identity map (of the spacetime) assigned to the Dirac evolution transport. Since these equations are equivalent to the Dirac equation (2.1), we call them *bundle (system of) Dirac equations*. The fact that now we have four equations, not a single one, reflects the flatness of U which, in its turn, reflects (due to the Heisenberg uncertainty relations) the non-existence of classical trajectories (world lines) for Dirac particles.

Now we shall introduce local bases and take a local view of the above-described.

Let $\{f_\mu(x)\}$ be a basis in $\mathcal{F}$ and $\{e_\mu(x)\}$ be a basis in F_x, $x \in M$. The matrices corresponding to vectors and/or linear maps (operators) in these fields of bases will be denoted by the same (kernel) symbol but in **boldface**, for instance: $\boldsymbol{\psi} := \big(\psi^0, \psi^1, \psi^2, \psi^3\big)^\top$ and $\boldsymbol{l}_x(y) := \big[(l_x)^\mu{}_\nu(y)\big]$ are defined, respectively, by $\psi(x) =: \psi^\mu(x) f_\mu(x)$ and $l_x\big(e_\nu(x)\big) =: \big(l_x(y)\big)^\mu{}_\nu f_\mu(y)$. We put $\boldsymbol{l}_x := \boldsymbol{l}_x(x)$; in fact this will be the only case when the matrix of l_x will be required as we want the 'physics in F_x' to correspond to that of $\mathcal{F}$ at $x \in M$. A very convenient choice is to put $e_\mu(x) = l_x^{-1}\big(f_\mu(x)\big)$; so then $\boldsymbol{l}_x = \mathbb{1}_4 := \big[\delta^\mu_\nu\big] = \mathrm{diag}(1, 1, 1, 1)$ is the 4×4 unit matrix.

The matrix $\boldsymbol{\mathcal{U}}(y, x)$ of $\mathcal{U}(y, x)$ is defined by

$$\boldsymbol{\psi}(y) =: \boldsymbol{\mathcal{U}}(y, x) \boldsymbol{\psi}(x) \tag{2.12}$$

and it is independent of ψ (see (2.2), (B.3), (B.4), and (B.5)). The matrix elements of $U(y, x)$ are defined via $U(y, x)\big(e_\mu(x)\big) =: U^\lambda{}_\mu(y, x) e_\lambda(y)$ and, due to (2.7), we have

$$\boldsymbol{U(y, x)} = \boldsymbol{l}_y^{-1} \cdot \boldsymbol{\mathcal{U}}(y, x) \cdot \boldsymbol{l}_x \tag{2.13}$$

which is generically, like $\boldsymbol{\mathcal{U}}$, a matrix operator (see Appendix B).

According to (A.12) the *coefficients* of the evolution transport form four matrix operators

$$
{}_{\mu}\boldsymbol{\Gamma}(x) := \big[{}_{\mu}\Gamma^{\lambda}{}_{\nu}(x) \big]_{\lambda,\nu=0}^{3} := \frac{\partial \boldsymbol{U}(x,y)}{\partial y^{\mu}}\bigg|_{y=x} = \boldsymbol{F}^{-1}(x) \frac{\partial \boldsymbol{F}(x)}{\partial x^{\mu}} \tag{2.14}
$$

where $\boldsymbol{U}(x,y) = \boldsymbol{F}^{-1}(y)\boldsymbol{F}(x)$ and $\boldsymbol{F}(x)$ is the matrix of a linear map (isomorphism) defining U according to theorem A.1.

Applying (A.5) and (2.14), we find

$$
\frac{\partial \boldsymbol{U}(y,x)}{\partial y^{\mu}} = - {}_{\mu}\boldsymbol{\Gamma}(y) \odot \boldsymbol{U}(y,x), \qquad \frac{\partial \boldsymbol{U}(y,x)}{\partial x^{\mu}} = \boldsymbol{U}(y,x) \odot {}_{\mu}\boldsymbol{\Gamma}(x), \tag{2.15}
$$

where $\odot$ denotes the introduced by (B.2) multiplication of matrix operators. Therefore

$$
\partial\!\!\!/_{y} \boldsymbol{U}(y,x) = -\boldsymbol{\Gamma}\!\!\!/(y)\, \boldsymbol{U}(y,x), \qquad \boldsymbol{\Gamma}\!\!\!/(x) := \gamma^{\mu} \cdot {}_{\mu}\boldsymbol{\Gamma}(x). \tag{2.16}
$$

Similarly to the nonrelativistic case [**?**], to any operator $\mathcal{A}\colon \mathcal{F} \to \mathcal{F}$ we assign a bundle morphism $A\colon F \to F$ by

$$
A_x := A|_{F_x} := l_x^{-1} \circ \mathcal{A} \circ l_x . \tag{2.17}
$$

Defining

$$
G^{\mu}(x) := l_x^{-1} \circ \gamma^{\mu} \circ l_x, \quad d_{\mu} := l_x^{-1} \circ \partial_{\mu} \circ l_x, \qquad \partial_{\mu} := \frac{\partial}{\partial x^{\mu}} \tag{2.18}
$$

and using the matrices $\mathcal{G}^{\mu}(x)$ and $\mathcal{E}_{\mu}(x)$ given via (B.12), we get

$$
\boldsymbol{G}^{\mu}(x) = \boldsymbol{l}_x^{-1} \mathcal{G}^{\mu}(x) \boldsymbol{l}_x, \quad \boldsymbol{d}_{\mu} := \mathbb{1}_4 \partial_{\mu} + \boldsymbol{l}_x^{-1} \big(\partial_{\mu} \boldsymbol{l}_x + \mathcal{E}_{\mu}(x) \boldsymbol{l}_x \big). \tag{2.19}
$$

(Here and below, for the sake of shortness, we sometimes omit the argument x.) The anticommutation relations

$$
\begin{gathered}
G^{\mu} G^{\nu} + G^{\nu} G^{\mu} = 2\eta^{\mu\nu} \,\mathsf{id}_F, \\
\boldsymbol{G}^{\mu} \boldsymbol{G}^{\nu} + \boldsymbol{G}^{\nu} \boldsymbol{G}^{\mu} = \mathcal{G}^{\mu}\mathcal{G}^{\nu} + \mathcal{G}^{\nu}\mathcal{G}^{\mu} = 2\eta^{\mu\nu} \mathbb{1}_4 \quad \big(= \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} \big),
\end{gathered} \tag{2.20}
$$

where $[\eta^{\mu\nu}] = \mathrm{diag}(1,-1,-1,-1) = [\eta_{\mu\nu}]$ is the Minkowski metric tensor, can be verified by means of (2.18), (2.19), (B.10), and the well know analogous relation for the γ-matrices (see, e.g. [**?**, chapter 2, equation (2.5)]).

For brevity, if $a_{\mu}\colon F \to F$ are morphisms, sums like $G^{\mu} \circ a_{\mu}$ will be denoted by 'backslashing' the kernel letter, $a\!\!\!\backslash := G^{\mu} \circ a_{\mu}$ (cf. the 'slashed' notation $a\!\!\!/ := \gamma^{\mu} a_{\mu}$). Similarly, we put $\boldsymbol{a}\!\!\!\backslash := \boldsymbol{G}^{\mu}(x) \boldsymbol{a}_{\mu}$ for $\boldsymbol{a}_{\mu} \in GL(4,\mathbb{C})$. It is almost evident (see (2.18)) that the morphism corresponding to $\partial\!\!\!/ := \gamma^{\mu}\partial_{\mu}$ is $d\!\!\!\backslash = G^{\mu}(x) \circ d_{\mu}$:

$$
d\!\!\!\backslash = l_x^{-1} \circ \partial\!\!\!/ \circ l_x . \tag{2.21}
$$

Using (2.19), we observe that

$$\begin{aligned}&\not{\boldsymbol{d}} = \not{\boldsymbol{\partial}} + \boldsymbol{G}^{\mu}(x)\boldsymbol{l}_x^{-1}\bigl(\partial_\mu \boldsymbol{l}_x + \boldsymbol{\mathcal{E}}_\mu(x)\boldsymbol{l}_x\bigr) = \not{\boldsymbol{\partial}} + \boldsymbol{G}^{\mu}(x)\boldsymbol{l}_x^{-1}(\partial_\mu \boldsymbol{l}_x) + \not{\boldsymbol{E}}(x),\\ &\boldsymbol{E}_\mu = \boldsymbol{l}_x^{-1}\boldsymbol{\mathcal{E}}_\mu \boldsymbol{l}_x.\end{aligned} \tag{2.22}$$

Now we are completely prepared for a more detailed exploration of Dirac equation (2.1) from bundle view-point.

First of all, we rewrite (2.1) as

$$\mathrm{i}\hbar\not{\partial}\psi = \mathcal{D}\psi, \qquad \mathcal{D} := mc\mathbb{1}_4 + \frac{e}{c}\not{\mathcal{A}}. \tag{2.23}$$

We claim that this is the *covariant* Schrödinger-like form of Dirac equation; $\not{\partial}$ is the analogue of the time derivation $\mathrm{d}/\mathrm{d}t$ and $\mathcal{D}$ corresponds to the Hamiltonian $\mathcal{H}$. We call $\mathcal{D}$ the *Dirac function*, or simply, *Diracian* of a particle described by Dirac equation.

Substituting (2.5) into (2.23), acting on the result from the left by l_x^{-1}, and using (2.18), we find the bundle form of (2.23) as

$$\mathrm{i}\hbar\not{d}\Psi = D\Psi \tag{2.24}$$

with $D \in \mathrm{Mor}(F,\pi,M)$ being the (Dirac) bundle morphism assigned to the Diracian. We call it *bundle Diracian.* According to (2.17) it is defined by

$$D_x := D|_{F_x} = l_x^{-1} \circ \mathcal{D} \circ l_x = mc\,\mathsf{id}_{F_x} + \frac{e}{c}\not{A}_x, \tag{2.25}$$

where $A_\mu = \mathcal{A}_\mu \,\mathsf{id}_F \in \mathrm{Mor}(F,\pi,M)$, $A_\mu|_x = l_x^{-1} \circ (\mathcal{A}_\mu|_x \,\mathsf{id}_{F_x}) \circ l_x = \mathcal{A}_\mu|_x \,\mathsf{id}_{F_x}$ are the components of the *bundle electromagnetic potential* and

$$\begin{aligned}\not{A}_x = G^{\mu}(x) \circ A_\mu|_x = G^{\mu}(x) \circ (l_x^{-1} \circ \mathcal{A}_\mu \,\mathsf{id}_{F_x} \circ l_x)&\\ = l_x^{-1} \circ (\not{\mathcal{A}}|_x \,\mathsf{id}_{F_x}) \circ l_x = \not{A}_x.&\end{aligned} \tag{2.26}$$

From (2.24), we get the straightforward bundle analogue of (2.23):[3]

$$\mathrm{i}\hbar\not{\boldsymbol{\partial}}\boldsymbol{\Psi} = \boldsymbol{D}^{\mathrm{m}}\boldsymbol{\Psi} \tag{2.27}$$

where (see (2.25))

$$\boldsymbol{D}^{\mathrm{m}} = \boldsymbol{D} - \mathrm{i}\hbar(\not{\boldsymbol{d}} - \not{\boldsymbol{\partial}}) = mc\mathbb{1}_4 + \frac{e}{c}\not{\boldsymbol{A}} - \mathrm{i}\hbar(\not{\boldsymbol{d}} - \not{\boldsymbol{\partial}}) \tag{2.28}$$

[3] We can not write (2.27) as $\mathrm{i}\hbar\not{\partial}\Psi = \cdots$ because the l.h.s. of such an equality will contain derivatives like $\partial_\mu\Psi(x) \equiv \partial\Psi(x)/\partial x^\mu$ which are not (well) defined at all due to $\Psi(x) \in F_x \neq F_y \ni \Psi(y)$ for $x, y \in M$ and $x \neq y$.

is the *matrix-bundle Diracian*[4] whose explicit form, due to (2.22), is

$$\boldsymbol{D}^{\mathrm{m}}\big|_{F_x} = mc\mathbb{1}_4 + \frac{e}{c}\slashed{\boldsymbol{A}}|_x - \mathrm{i}\hbar\big(\boldsymbol{G}^{\mu}(x)\boldsymbol{l}_x^{-1}(\partial_\mu \boldsymbol{l}_x) + \slashed{\boldsymbol{E}}\big), \qquad \boldsymbol{E}_\mu = \boldsymbol{l}_x^{-1}\boldsymbol{\mathcal{E}}_\mu(x)\boldsymbol{l}_x. \tag{2.29}$$

The matrix-bundle Diracian is closely connected with the matrices (2.14) formed from the coefficients of the Dirac evolution transport. Applying the formalism developed, we can establish this connection by two independent ways, leading, of course, to one and the same final result.

The first way is to insert (2.6) into (2.24) and acting on the result from the left by l_x^{-1}, we get the bundle version of the initial-value problem (2.3) (see (2.24) and (2.10)):

$$\mathrm{i}\hbar\slashed{\partial} \circ U(x,x_0) = D \circ U(x,x_0), \quad U(x_0,x_0) = \mathsf{id}_{F_{x_0}}, \tag{2.30}$$

or

$$\mathrm{i}\hbar\slashed{\boldsymbol{\partial}} \odot \boldsymbol{U}(x,x_0) = \boldsymbol{D}^{\mathrm{m}} \odot \boldsymbol{U}(x,x_0), \quad \boldsymbol{U}(x_0,x_0) = \mathbb{1}_4. \tag{2.30$'$}$$

Since (2.15) implies (cf. (2.16))

$$\slashed{\partial}_y \boldsymbol{U}(y,x) = -\slashed{\boldsymbol{\Gamma}}(y) \odot \boldsymbol{U}(y,x), \tag{2.31}$$

$$\slashed{\boldsymbol{\Gamma}}(x) := \boldsymbol{G}^{\mu}(x) \cdot {}_{\mu}\boldsymbol{\Gamma}(x), \tag{2.32}$$

from (**??**$'$) we conclude

$$\slashed{\boldsymbol{\Gamma}}(x) = -\frac{1}{\mathrm{i}\hbar}\boldsymbol{D}^{\mathrm{m}}(x) \quad \Big(= -\frac{1}{\mathrm{i}\hbar}(mc\mathbb{1}_4 + \frac{e}{c}\slashed{\boldsymbol{A}}) + \boldsymbol{G}^{\mu}\boldsymbol{l}_x^{-1}(\partial \boldsymbol{l}_x) + \slashed{\boldsymbol{E}}\Big). \tag{2.33}$$

So, the *'backslashed' matrix of the coefficients of the Dirac evolution transport coincides up to a constant with the matrix-bundle Diracian.*[5]

The another way mentioned for obtaining (2.33) is to write (2.11) into a matrix form, which by virtue of (A.11) is

$$\partial_\mu \boldsymbol{\Psi} + {}_{\mu}\boldsymbol{\Gamma} \cdot \boldsymbol{\Psi} = 0, \tag{2.34}$$

then multiplying from the left by $G^{\mu}(x)$, summing over μ, and comparing the result with (2.27), we get (2.33).

From the first derivation of (2.33) one can easily see that, in fact, the equation (2.33) is equivalent to (2.30) and, consequently, to the initial Dirac equation (2.1). Hence (2.33) is an *algebraic bundle* form of the *Dirac equation.*

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[4]The matrix operator $\boldsymbol{D}^{\mathrm{m}}$ is the relativistic analogue of the matrix-bundle Hamiltonian [**?**, sect. 2].

[5]Cf. an analogous result connecting the matrix of the coefficients of the evolution transport and the matrix-bundle Hamiltonian in bundle nonrelativistic quantum mechanics (see [**?**, equation (2.21] or (I.2.10)).

Here the following natural problem arises. As we have just proved, (2.33) is equivalent to the Dirac equation (2.1). So, if $\not{\boldsymbol{\Gamma}}$ is given (by (2.33)), we can recover the transport U, but it is known [?, ?, ?] that U can uniquely be reconstructed by the matrices (2.14). Hence the *four* matrices (2.14) have to be equivalent to the *one* matrix (2.32). Is this possible? The answer is positive.[6] Actually, substituting (2.14) into (2.32) and taking into account (2.33), we observe that[7]

$$\mathrm{i}\hbar\not{\boldsymbol{\partial}}\boldsymbol{F}^{-1} = \boldsymbol{D}^{\mathrm{m}}\boldsymbol{F}^{-1} \quad (= -\mathrm{i}\hbar\not{\boldsymbol{\Gamma}}\boldsymbol{F}^{-1}). \tag{2.35}$$

So, given $\not{\boldsymbol{\Gamma}}$ and taking $\boldsymbol{F}$ to be any solution of the matrix Dirac equation (2.35) (with respect to $\boldsymbol{F}^{-1}$), we can calculate ${}_{\mu}\boldsymbol{\Gamma}$ via (2.14). On the opposite, given ${}_{\mu}\boldsymbol{\Gamma}$ or F, then $\not{\boldsymbol{\Gamma}} := \boldsymbol{G}^{\mu}\,{}_{\mu}\boldsymbol{\Gamma} = -\big(\not{\partial}\boldsymbol{F}^{-1}\big)\boldsymbol{F}$.

3. Other relativistic wave equations

The relativistic-covariant Klein-Gordon equation for a (spinless) particle of mass m and electric charge e in a presence of (external) electromagnetic field with 4-potential $\mathcal{A}_\mu$ is [?, chapter XX, § 5, equation (30′)]

$$\Big(\mathcal{D}^{\mu}\mathcal{D}_{\mu} + \frac{m^2c^2}{\hbar^2}\Big)\phi = 0, \qquad \mathcal{D}_\mu = \eta_{\mu\nu}\mathcal{D}^{\nu} := \partial_\mu - \frac{e}{\mathrm{i}\hbar c}\mathcal{A}_\mu. \tag{3.1}$$

Since this is a second-order partial differential equation, it does not directly admit an evolution operator and adequate bundle formulation and interpretation. To obtain such a formulation, we have to rewrite (3.1) as a first-order (system of) partial differential equation(s) (cf. Sect. I.5).

Perhaps the best way to do this is to replace ϕ with a 5×1 matrix $\varphi = (\varphi^0,\dots,\varphi^4)^\top$ and to introduce 5×5 Γ-matrices Γ^μ, $\mu = 0,1,2,3$ with components $\big(\Gamma^\mu\big)^i{}_j$, $i,j = 0,1,2,3,4$ such that (cf. [?, chapter I, equations (4.38) and (4.37)]):

$$\varphi = a\begin{pmatrix}\mathrm{i}\hbar\mathcal{D}_0\phi\\ \mathrm{i}\hbar\mathcal{D}_1\phi\\ \mathrm{i}\hbar\mathcal{D}_2\phi\\ \mathrm{i}\hbar\mathcal{D}_3\phi\\ mc\phi\end{pmatrix}, \qquad \big(\Gamma^\mu\big)^i{}_j = \begin{cases}1 & \text{for } (i,j) = (\mu,4)\\ \eta_{\mu\mu} & \text{for } (i,j) = (4,\mu)\\ 0 & \text{otherwise}\end{cases} \tag{3.2}$$

where the complex constant $a \neq 0$ is insignificant for us and can be (partially) fixed by an appropriate normalization of φ.

[6] The real reason for that fact is the flatness of the Dirac transport U, i.e., for instance, the fact that $U(y,x)$ depends only on the initial and final points x and y, respectively, and not on other objects as some path (curve) connecting them. All this reflects the nonexistence of (classical) trajectories of quantum particles.

[7] In fact (2.35) is the integrability condition for (2.14) considered as a first-order system of partial differential equations with respect to $\boldsymbol{F}(x)$.

Then a simple checking shows that (3.1) is equivalent to (cf. (2.1))

$$(\mathrm{i}\hbar\Gamma^{\mu}\mathcal{D}_{\mu} - mc\mathbb{1}_5)\varphi = 0 \tag{3.3}$$

with $\mathbb{1}_5 = \operatorname{diag}(1,1,1,1,1)$ being the 5×5 unit matrix, or (cf. (2.23))

$$\mathrm{i}\hbar\Gamma^{\mu}\partial_{\mu}\varphi = \mathcal{K}\varphi, \qquad \mathcal{K} := mc\mathbb{1}_5 + \frac{e}{c}\Gamma^{\mu}\mathcal{A}_{\mu}. \tag{3.4}$$

Now it is evident that *mutatis mutandis*, taking Γ^{μ} for γ^{μ}, φ for ψ, etc., all the (bundle) machinery developed in Sect. 2 for the Dirac equation can be applied to the Klein-Gordon equation in the form (3.3) practically without changes. Since the transferring of the results obtained in Sect. 2 for Dirac equation to Klein-Gordon one is absolutely trivial,[8] we are not going to present here the bundle description of the latter equation.

Since the relativistic wave equations for particles with spin greater than 1/2 are versions or combinations of Dirac and Klein-Gordon equations [**?**, **?**, **?**, **?**], for them is *mutatis mutandis* applicable the bundle approach developed in Sect. 2 for Dirac equation or/and its version for Klein-Gordon one pointed above.

4. Propagators and evolution transports or operators

The propagators, called also propagator functions or Green functions, are solutions of the wave equations with point-like unit source and satisfy appropriate (homogeneous) boundary conditions corresponding to the concrete problem under exploration [**?**, **?**]. Undoubtedly these functions play an important rôle in the mathematical apparatus of (relativistic) quantum mechanics and its physical interpretation [**?**, **?**]. By this reason it is essential to be investigated the connection between propagators and evolution operators or/and transports. As we shall see below, the latter can be represented as integral operators whose kernel is connected in a simple way with the corresponding propagator.

4.1. Green functions (review)

Generally [**?**, article "Green function"] the *Green function* $g(x', x)$ of a linear differential operator L (or of the equation $Lu(x) = f(x)$) is the kernel of the integral operator inverse to L. As the kernel of the unit operator is the Dirac delta-function $\delta^4(x'-x)$, the Green function is a fundamental solution of the non-homogeneous equation

$$Lu(x) = f(x), \tag{4.1}$$

[8]Both coincide up to notation or a meaning of the corresponding symbols.

i.e. treated as a generalized function $g(x',x)$ is a solution of

$$L_{x'} g(x',x) = \delta^4(x'-x). \tag{4.2}$$

Given the Geen function $g(x',x)$, the solution of (4.1) is

$$u(x') = \int g(x',x) f(x) \, \mathrm{d}^4 x. \tag{4.3}$$

A concrete Green function $g(x',x)$ for L (or (4.1)) satisfies, besides (4.2), certain (homogeneous) boundary conditions on x' with fixed x, i.e. it is the solution of a fixed boundary-value problem for equation (4.2). Hence, if $g_f(x',x)$ is some fundamental solution, then

$$g(x',x) = g_f(x',x) + g_0(x',x) \tag{4.4}$$

where $g_0(x',x)$ is a solution of the homogeneous equation $L_{x'} g_0(x',x) = 0$ chosen such that $g(x',x)$ satisfies the required boundary conditions.

Suppose $g(x',x)$ is a Green function of L for some boundary-value (or initial-value) problem. Then, using (4.2), we can verify that

$$L_{x'} \Big(\int g(x',x) u(x) \, \mathrm{d}^3 \boldsymbol{x} \Big) = \delta(ct'-ct) u(x') \tag{4.5}$$

where $x = (ct, \boldsymbol{x})$ and $x' = (ct', \boldsymbol{x}')$. Therefore the solution of the problem

$$L_x u(x) = 0, \quad u(ct_0, \boldsymbol{x}) = u_0(\boldsymbol{x}) \tag{4.6}$$

is

$$u(x) = \int g\big(x, (ct_0, \boldsymbol{x}_0)\big) u_0(\boldsymbol{x}_0) \, \mathrm{d}^3 \boldsymbol{x}_0, \quad x_0 = (ct_0, \boldsymbol{x}_0) \qquad \text{for } t \neq t_0. \tag{4.7}$$

From here we can make the conclusion that, if (4.6) admits an evolution operator $\mathcal{U}$ such that (cf. (I.2.2))

$$u(x) \equiv u(ct, \boldsymbol{x}) = \mathcal{U}(t, t_0) u(ct_0, \boldsymbol{x}) \tag{4.8}$$

or (cf. (2.2))

$$u(x) = \mathcal{U}(x, x_0) u(x_0), \tag{4.9}$$

then the r.h.s of (4.7) realizes $\mathcal{U}$ as an integral operator with a kernel equal to the Green function g.

Since all (relativistic or not) wave equations are versions of (4.6), the corresponding evolution operators, if any, and Green functions (propagators) are connected as just described. Moreover, if some wave equation does not admit (directly) evolution operator, e.g. if it is of order greater than one, then we can *define* it as the corresponding version of the integral operator in the r.h.s. of (4.7). In this way is established a one-to-one onto correspondence between the evolution operators and Green functions for any particular problem like (4.6).

And a last general remark. The so-called S-matrix finds a lot of applications in quantum theory [**?**,**?**,**?**]. By definition S is an operator transforming the system's state vector $\psi(-\infty, \boldsymbol{x})$ before scattering (reaction) into the one $\psi(+\infty, \boldsymbol{x})$ after it:

$$\lim_{t\to+\infty} \psi(ct, \boldsymbol{x}) =: S \lim_{t\to-\infty} \psi(ct, \boldsymbol{x}).$$

So, e.g. when (4.8) takes place, we have

$$S = \lim_{t_\pm\to\pm\infty} \mathcal{U}(t_+, t_-) =: \mathcal{U}(+\infty, -\infty). \tag{4.10}$$

Thus the above-mentioned connection between evolution operators and Green functions can be used for expressing the S-matrix in terms of propagators. Such kind of formulae are often used in relativistic quantum mechanics [**?**].

4.2. Nonrelativistic case (Schrödinger equation)

The (retarded) Green function $g(x', x)$, $x', x \in M$, for the Schrödinger equation (I.2.1) is defined as the solution of the boundary-value problem [**?**, § 22]

$$\big[\mathrm{i}\hbar\frac{\partial}{\partial t'} - \mathcal{H}(x')\big] g(x', x) = \delta^4(x' - x), \tag{4.11}$$

$$g(x', x) = 0 \qquad \text{for } t' < t \tag{4.12}$$

where $x' = (ct', \boldsymbol{x}')$ $x = (ct, \boldsymbol{x})$, $\mathcal{H}(x)$ is system's Hamiltonian, and $\delta^4(x' - x)$ is the 4-dimensional (Dirac) δ-function.

Given g, the solution $\psi(x')$ (for $t' > t$) of (I.2.1) is[9]

$$\theta(t' - t)\psi(x') = \mathrm{i}\hbar \int \mathrm{d}^3\boldsymbol{x} g(x', x)\psi(x) \tag{4.13}$$

where the θ-function $\theta(s)$, $s \in \mathbb{R}$ is defined by $\theta(s) = 1$ for $s > 0$ and $\theta(s) = 0$ for $s < 0$.

[9]One should not confuse the notation $\psi(x) = \psi(ct, \boldsymbol{x})$, $x = (ct, \boldsymbol{x})$ of this section and $\psi(t)$ from Sect. I.2. The latter is the wavefunction at a moment t and the former is its value at the spacetime point $x = (ct, \boldsymbol{x})$. Analogously, $\Psi_\gamma(x) \equiv \Psi_\gamma(ct, \boldsymbol{x}) := l^{-1}_{\gamma(t)}\big(\psi(ct, \boldsymbol{x})\big)$ should not be confused with $\Psi_\gamma(t)$ from Sect. I.2. A notation like $\psi(t)$ and $\Psi_\gamma(t)$ will be used if the spatial parts of the arguments is inessential, as in Sect. I.2, and there is no risk of ambiguities.

Combining (I.2.2) and (4.13), we find the basic connection between the evolution operator $\mathcal{U}$ and Green function of Schrödinger equation:

$$\theta(t'-t)\big[\mathcal{U}(t',t)\big(\psi(ct,\boldsymbol{x}')\big)\big] = \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{x}\, g\big((ct',\boldsymbol{x}'),(ct,\boldsymbol{x})\big)\psi(ct,\boldsymbol{x}). \quad (4.14)$$

Actually this formula, if g is known, determines $\mathcal{U}(t',t)$ for all t' and t, not only for $t'>t$, as $\mathcal{U}(t',t)=\mathcal{U}^{-1}(t,t')$ and $\mathcal{U}(t,t)=\mathsf{id}_{\mathcal{F}}$ with $\mathcal{F}$ being the system's Hilbert space (see (I.2.2) or [**?**, sect. 2]). Consequently *the evolution operator for the Schrödinger equation can be represented as an integral operator whose kernel up to the constant* i$\hbar$ *is exactly the (retarded) Green function for it.*

To write the bundle version of (4.14), we introduce the *Green operator* which is simply a multiplication with the Green function:

$$\mathcal{G}(x',x) := g(x',x)\,\mathsf{id}_{\mathcal{F}}\colon \mathcal{F}\to\mathcal{F}. \quad (4.15)$$

The corresponding to it *Green morphism* G is given via (see (I.2.11) and cf. (I.2.4))

$$G_\gamma(x',x) := l^{-1}_{\gamma(t')}\circ\mathcal{G}(x',x)\circ l_{\gamma(t)} = g(x',x)l^{-1}_{\gamma(t')}\circ l_{\gamma(t)}\colon F_{\gamma(t)}\to F_{\gamma(t')}. \quad (4.16)$$

Now, acting on (4.14) from the left by $l^{-1}_{\gamma(t')}$ and using (I.2.3) and (I.2.4), we obtain

$$\theta(t'-t)\big[\,U_\gamma(t',t)\big(\Psi_\gamma(ct,\boldsymbol{x}')\big)\big] = \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{x}\, G_\gamma\big((ct',\boldsymbol{x}'),(ct,\boldsymbol{x})\big)\Psi_\gamma(ct,\boldsymbol{x}). \quad (4.17)$$

Therefore for the Schrödinger equation the *the evolution transport U can be represented as an integral operator with kernel equal to* i$\hbar$ *times the Green morphism G.*

Taking as a starting point (4.14) and (4.17), we can obtain different representations for the evolution operator and transport by applying concrete formulae for the Green function. For example, if a complete set $\{\psi_a(x)\}$ of orthonormal solutions of Schrödinger equation satisfying the completeness condition[10]

$$\sum_a \psi_a(ct,\boldsymbol{x}')\psi^*_a(ct,\boldsymbol{x}) = \delta^3(\boldsymbol{x}'-\boldsymbol{x})$$

is know, then [**?**, § 22]

$$g(x',x) = \frac{1}{\mathrm{i}\hbar}\theta(t'-t)\sum_a \psi_a(x')\psi^*_a(x)$$

[10]Here and below the symbol $\sum_a$ denotes a sum and/or integral over the discrete and/or continuous spectrum. The asterisk $(*)$ means complex conjugation.

which, when substituted into (4.14), implies

$$\mathcal{U}(t',t)\psi(ct,\boldsymbol{x}') = \sum_a \psi_a(x') \int \mathrm{d}^3\, x \psi_a^*(ct,\boldsymbol{x})\psi(ct,\boldsymbol{x}). \tag{4.18}$$

Note, the integral in this equation is equal to the a-th coefficient of the expansion of ψ over $\{\psi_a\}$.

4.3. Dirac equation

Since the Dirac equation (2.1) is a first-order linear partial differential equation, it admits both evolution operator and Green function(s) (propagator(s)). From a generic view-point, the only difference from the Schrödinger equation is that (2.1) is a *matrix* equation; so the corresponding Green functions are actually Green matrices, i.e. Green matrix-valued functions. Otherwise the results of Subsect. 4.2 are *mutatis mutandis* transferred into the theory of Dirac equation.

The (retarded) Green matrix (function) or propagator for Dirac equation (2.1) is a 4×4 matrix-valued function $g(x',x)$ depending on two arguments $x', x \in M$ and such that [**?**, sect. 2.5.1 and 2.5.2]

$$(\mathrm{i}\hbar \not{D}_{x'} - mc\mathbb{1}_4) g(x',x) = \delta(x'-x) \tag{4.19}$$

$$g(x',x) = 0 \qquad \text{for } t < t'. \tag{4.20}$$

For a free Dirac particle, i.e. for $\not{D} = \not{\partial}$ or $\mathcal{A}_\mu = 0$, the explicit expression $g_0(x',x)$ for $g(x',x)$ is derived in [**?**, sect. 2.5.1], where the notation $\mathcal{K}$ instead of g_0 is used. In an external electromagnetic field $\mathcal{A}_\mu$ the Green matrix g is a solution of the integral equation[11]

$$g(x',x) = g_0(x',x) + \int \mathrm{d}^4 y\, g_0(x',y) \frac{e}{c} \not{\mathcal{A}}(y) g(y,x) \tag{4.21}$$

which includes the corresponding boundary condition.[12] The iteration of this equation results in the perturbation series for g (cf. [**?**, sect. 2.5.2]).

If the (retarded) Green matrix $g(x',x)$ is known, the solution $\psi(x')$ of Dirac equation (for $t' > t$) is

$$\theta(t'-t)\psi(x') = \mathrm{i}\hbar \int \mathrm{d}^3\boldsymbol{x}\, g(x',x)\gamma^0\psi(x). \tag{4.22}$$

[11] The derivation of (4.21) is the same as for the Feynman propagators S_F and S_A given in [**?**, sect. 2.5.2]. The propagators S_F and S_A correspond to g_0 and g respectively, but satisfy other boundary conditions [**?**, **?**].

[12] Due to (4.4), the integral equation (4.21) is valid for any Green function (matrix) of the Dirac equation (2.1).

Hence, denoting by $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ the non-relativistic (see Sect. I.4) and relativistic (see Sect. 2), respectively, Dirac evolution operators, from the equations (I.2.2), (2.2), and (4.22), we find:

$$\begin{aligned}\theta(t'-t)\big[\widetilde{\mathcal{U}}(t',t)\big(\psi(ct,\boldsymbol{x}')\big)\big] &= \theta(t'-t)\big[\mathcal{U}(x',x)\big(\psi(x)\big)\big] \\ &= \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{x}\, g\big((ct',\boldsymbol{x}'),(ct,\boldsymbol{x})\big)\gamma^0\psi(ct,\boldsymbol{x}). \qquad (4.23)\end{aligned}$$

So, in both cases the evolution operator admits an integral representation whose kernel up to the right multiplication with $\mathrm{i}\hbar\gamma^0$ is equal to the (retarded) Green function for Dirac equation.

Similarly to (4.17), now the bundle version of (4.23) is

$$\theta(t'-t)\big[\widetilde{U}(t',t)\big(\Psi_\gamma(ct,\boldsymbol{x}')\big)\big] = \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{x}\; G_\gamma(x',x)G^0(\gamma(t))\Psi_\gamma(ct,\boldsymbol{x}), \qquad (4.24a)$$

$$\theta(t'-t)\big[U(x',x)\big(\Psi(x)\big)\big] = \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{x}\; G(x',x)G^0(x)\Psi(x) \qquad (4.24b)$$

where $G^0(x)$ is defined in (2.18) for $\mu = 0$ and (cf. (4.16))

$$G_\gamma(x',x) := l^{-1}_{\gamma(t')}\circ\mathcal{G}(x',x)\circ l_{\gamma(t)}, \quad G(x',x) := l^{-1}_{x'}\circ\mathcal{G}(x',x)\circ l_x \qquad (4.25)$$

with (cf. . (4.15))

$$\mathcal{G}(x',x) = g(x',x)\,\mathsf{id}_{\mathcal{F}}. \qquad (4.26)$$

(Here $\mathcal{F}$ is the space of 4-spinors.)

Analogously to the above results, one can obtained such for other propagators, e.g. for the Feynman one [**?**, **?**], but we are not going to do this here as it is a trivial variant of the procedure described.

4.4. Klein-Gordon equation

The (retarded) Green function $g(x',x)$ for the Klein-Gordon equation (3.1) is a solution to the boundary-value problem [**?**, sect. 1.3.1]

$$\Big(\mathcal{D}^\mu\mathcal{D}_\mu + \frac{m^2c^2}{\hbar^2}\Big)\Big|_{x'} g(x',x) = \delta^4(x'-x), \qquad (4.27)$$

$$g(x',x) = 0 \qquad \text{for } t' < t. \qquad (4.28)$$

For a free particle its explicit form can be found in [**?**, sect. 1.3.1].

A simple verification proves that, if $g(x',x)$ is known, the solution ϕ of (3.1) (for $t' > t$) is given by ($x^0 = ct$)

$$\theta(t'-t)\phi(x') = \int \mathrm{d}^3\boldsymbol{x}\,\Big[\frac{\partial g(x',x)}{\partial x^0}\phi(x) + g(x',x)\Big(2\frac{\partial\phi(x)}{\partial x^0} - \frac{e}{\mathrm{i}\hbar c}\mathcal{A}^0(x)\phi(x)\Big)\Big]. \qquad (4.29)$$

Introducing the matrices

$$\psi(x) := \begin{pmatrix} \phi(x) \\ \partial_0|_x \phi(x) \end{pmatrix}, \qquad \mathsf{g}(x',x) := \begin{pmatrix} \mathcal{D}_0|_x g(x',x) \\ 2g(x',x) \end{pmatrix}, \tag{4.30}$$

we can rewrite (4.29) as

$$\theta(t'-t)\phi(x') = \int \mathrm{d}^3\boldsymbol{x}\, \mathsf{g}^\top(x',x)\cdot\psi(x) \tag{4.31}$$

where the dot $(\cdot)$ denotes matrix multiplication.

An important observation is that for ψ the Klein-Gordon equation transforms into first-order Schrödinger-type equation (see Sect. I.5) with Hamiltonian ${}^{\mathrm{K-G}}_{\quad\mathrm{c}}\mathcal{H}$ given by (I.5.3) in which $\mathsf{id}_{\dots}$ is replaced by $c\,\mathsf{id}_{\dots}$.

Denoting the (retarded) Green function, which is in fact 2×2 matrix, and the evolution operator for this equation by $\widetilde{\mathcal{G}}(x',x) = \bigl[\widetilde{\mathcal{G}}^a{}_b(x',x)\bigr]^2_{a,b=1}$ and $\widetilde{\mathcal{U}}(t',t) = \bigl[\widetilde{\mathcal{U}}^a{}_b(t',t)\bigr]^2_{a,b=1}$, respectively, we see that (cf. Subsect. 4.2, equation (4.13))

$$\theta(t'-t)\bigl[\widetilde{\mathcal{U}}(t',t)\psi(ct,\boldsymbol{x})\bigr] = \theta(t'-t)\psi(x') = \int \mathrm{d}^3\boldsymbol{x}\,\widetilde{\mathcal{G}}(x',x)\psi(x). \tag{4.32}$$

Comparing this equations with (4.30) and (4.31), we find

$$\bigl(\widetilde{\mathcal{G}}^1{}_1(x',x), \widetilde{\mathcal{G}}^1{}_2(x',x)\bigr) = \mathsf{g}^\top(x',x).$$

The other matrix elements of $\widetilde{\mathcal{G}}$ can also be connected with $\mathsf{g}(x',x)$ and its derivatives, but this is inessential for the following.

In this way we have connected, via (4.32), the evolution operator and the (retarded) Green function for a concrete first-order realization of Klein-Gordon equation. It is almost evident that this procedure *mutatis mutandis* works for any such realization; in every case the corresponding Green function (resp. matrix) being a (resp. matrix-valued) function of the Green function $g(x',x)$ introduce via (4.27) and (4.28). For instance, the treatment of the 5-dimensional realization given by (3.2) and (3.3) is practically identical to the one of Dirac equation in Subsect. 4.3, only the γ-matrices γ^μ have to be replace with the 5×5 matrices Γ^μ (defined by (3.2)). This results into a 5×5 matrix evolution operator $\mathcal{U}(x',x)$, etc.

Since the bundle version of (4.32) or an analogous result for $\mathcal{U}(x',x)$ is absolutely trivial (cf. Subsect. 4.2 and 4.3 resp.), we are not going to write it here; up to the meaning of notation it coincides with (4.17) or (4.24b) respectively.

At the end, we notice that via (4.29) can be defined an 'evolution operator' $\phi(x)\mapsto\phi(x')$ (or $\phi(ct,\boldsymbol{x})\mapsto\phi(ct',\boldsymbol{x})$), but such a map does not have 'good' properties and does not find applications.

5. Conclusion

In this investigation we have reformulated the relativistic wave equations in terms of fibre bundles. In the bundle formulation the wavefunctions are represented as (state) liftings of paths or sections along paths (time-dependent approach) or simply sections (covariant approach) of a suitable vector bundle over the spacetime. The covariant approach, developed in the present work, has an advantage of being explicitly covariant while in the time-dependent one the time plays a privilege rôle. In both cases the evolution (in time or in spacetime resp.) is described via an evolution transport which is a linear transport (along paths or along the identity map of the spacetime resp.) in the bundle mentioned. The state liftings or sections are linearly transported by means of the corresponding evolution transports. An equivalent version of this statement is the new bundle form of the wave equations: the derivation(s) assigned to the evolution transports of the corresponding state sections identically vanish. We also have explored some links between evolution operators or transports and the retarded Green functions (or matrices) for the corresponding wave equations: the former turn to have realization as integral operators whose kernel is equal to the latter ones up to a multiplication with a constant complex number or matrix.

These connections suggest the idea for introducing 'retarded', or, in a sense, 'causal' evolution operators or transports as a product of the evolution operators or transports with θ-function of the difference of the times corresponding to the first and second arguments of the transport or operator.

A further development of the ideas presented in this investigation naturally leads to their application to (quantum) field theory which will be done elsewhere.

Appendix A. Linear transports along maps in fibre bundles

In this appendix we recall a few simple facts concerning (linear) transports along maps, in particular along paths, required for the present investigation. The below-presented material is abstracted from [**?**,**?**,**?**] where further details can be found (see also [**?**, sect. 3]).

Let (E, π, B) be a topological bundle with base B, bundle (total, fibre) space E, projection $\pi : E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$. Let the set N be not empty, $N \neq \varnothing$, and there be given a map $\varkappa : N \to B$. By id_X is denoted the identity map of a set X.

Definition A.1. A transport along maps in the bundle (E, π, B) is a map K assigning to any map $\varkappa : N \to B$ a map $K^{\varkappa}$, transport along $\varkappa$, such that $K^{\varkappa}\colon (l,m)\mapsto K^{\varkappa}_{l\to m}$, where for every $l,m\in N$ the map

$$K^{\varkappa}_{l\to m}\colon \pi^{-1}(\varkappa(l))\to\pi^{-1}(\varkappa(m)), \tag{A.1}$$

called transport along $\varkappa$ from l to m, satisfies the equalities:

$$K^{\varkappa}_{m\to n}\circ K^{\varkappa}_{l\to m}=K^{\varkappa}_{l\to n}, \qquad l,m,n\in N, \tag{A.2}$$
$$K^{\varkappa}_{l\to l}=\mathsf{id}_{\pi^{-1}(\varkappa(l))}, \qquad l\in N. \tag{A.3}$$

If (E,π,B) is a complex (or real) vector bundle and the maps (A.1) are linear, i.e.

$$K^{\varkappa}_{l\to m}(\lambda u+\mu v)=\lambda K^{\varkappa}_{l\to m}u+\mu K^{\varkappa}_{l\to m}v,\quad \lambda,\mu\in\mathbb{C}\ (\text{or }\mathbb{R}),\quad u,v\in\pi^{-1}(\varkappa(l)), \tag{A.4}$$

the transport K is called *linear*. If $\varkappa$ belongs to the set of paths in B, $\varkappa\in\{\gamma\colon J\to B,\ J \text{ being } \mathbb{R}\text{-interval}\}$, we said that the transport K is *along paths.*

For the present work is important that the class of linear transports along the identity map id_B of B coincides with the class of *flat* linear transports along paths[13] (see the comments after equation (2.3) of [**?**]).

The general form of a transport along maps is described by the following result.

Theorem A.1. *Let $\varkappa\colon N\to B$. The map $K\colon\varkappa\mapsto K^{\varkappa}\colon(l,m)\mapsto K^{\varkappa}_{l\to m}$, $l,m\in N$ is a transport along $\varkappa$ if and only if there exist a set Q and a family of bijective maps $\{F^{\varkappa}_n\colon\pi^{-1}(\varkappa(n))\to Q,\ n\in N\}$ such that*

$$K^{\varkappa}_{l\to m}=(F^{\varkappa}_m)^{-1}\circ(F^{\varkappa}_l),\quad l,m\in N. \tag{A.5}$$

The maps $F^{\varkappa}_n$ are defined up to a left composition with bijective map depending only on $\varkappa$, i.e. (A.5) *holds for given families of maps $\{F^{\varkappa}_n\colon\pi^{-1}(\varkappa(n))\to Q,\ n\in N\}$ and $\{{}'F^{\varkappa}_n\colon\pi^{-1}(\varkappa(n))\to{}'Q,\ n\in N\}$ for some sets Q and ${}'Q$ iff there is a bijective map $D^{\varkappa}\colon Q\to{}'Q$ such that*

$${}'F^{\varkappa}_n=D^{\varkappa}\circ F^{\varkappa}_n,\quad n\in N. \tag{A.6}$$

For the purposes of this investigation we need a slight generalization of [**?**, definition 4.1], viz we want to replace in it $\mathbf{N}\subset\mathbb{R}^k$ with an arbitrary differentiable manifold. Let N be a differentiable manifold and $\{x^a : a=1,\dots,\dim N\}$ be coordinate system in a neighborhood of $l\in N$. For $\varepsilon\in(-\delta,\delta)\subset\mathbb{R}$, $\delta\in\mathbb{R}_+$ and $l\in N$ with coordinates $l^a=x^a(l)$, we define

[13] The *flat* linear transports along paths are defined as ones with vanishing curvature operator [**?**, sect. 2]. By [**?**, theorem. 6.1] we can equivalently define them by the property that they depend only on their initial and final points, i.e. if $K^{\gamma}_{s\to t}$, $\gamma\colon J\to B$, $s,t\in J\subset\mathbb{R}$, depends only on $\gamma(s)$ and $\gamma(t)$ but not on the path γ itself.

$l_b(\varepsilon) \in N$, $b = 1, \dots, \dim N$ by $l_b^a(\varepsilon) := x^a(l_b(\varepsilon)) := l^a + \varepsilon\delta_b^a$ where the Kroneker δ-symbol is given by $\delta_b^a = 0$ for $a \neq b$ and $\delta_b^a = 1$ for $a = b$. Let $\xi = (E, \pi, B)$ be a vector bundle, $\varkappa\colon N \to B$ be injective (i.e. 1:1 mapping), and $\mathrm{Sec}^p(\xi)$ (resp. $\mathrm{Sec}(\xi)$) be the set of C^p (resp. all) sections over ξ. Let $L^{\varkappa}_{l\to m}$ be a C^1 (on l) *linear* transport along $\varkappa$. Now the modified definition reads:[14]

Definition A.2. The a-th, $1 \leq a \leq \dim N$ partial (section-)derivation along maps generated by L is a map ${}_a\mathcal{D} : \varkappa \mapsto {}_a\mathcal{D}^{\varkappa}$ where the a-th (partial) derivation ${}_a\mathcal{D}^{\varkappa}$ along $\varkappa$ (generated by L) is a map

$$ {}_a\mathcal{D}^{\varkappa} : l \mapsto \mathcal{D}^{\varkappa}_{l^a}, \tag{A.7}$$

where the a-th (partial) derivative $\mathcal{D}^{\varkappa}_{l^a}$ along $\varkappa$ at l is a map

$$ \mathcal{D}^{\varkappa}_{l^a} : \mathrm{Sec}^1\left(\xi|_{\varkappa(N)}\right) \to \pi^{-1}(\varkappa(l)) \tag{A.8}$$

defined for $\sigma \in \mathrm{Sec}^1\left(\xi|_{\varkappa(N)}\right)$ by

$$ \left({}_a\mathcal{D}^{\varkappa}\sigma\right)(\varkappa(l)) := \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left(L^{\varkappa}_{l_a(\varepsilon)\to l}\sigma(\varkappa(l_a(\varepsilon))) - \sigma(\varkappa(l))\right)\right]. \tag{A.9}$$

Accordingly can be modified the other definitions of [**?**, sect. 4], all the results of it being *mutatis mutandis* valid. In particular, we have:

Proposition A.1. *The operators $\mathcal{D}^{\varkappa}_{l^a}$ are ($\mathbb{C}$-)linear and*

$$ \mathcal{D}^{\varkappa}_{m^a} \circ L^{\varkappa}_{l\to m} \equiv 0. \tag{A.10}$$

Proposition A.2. *If $\sigma \in \mathrm{Sec}^1\left(\xi|_{\varkappa(N)}\right)$, then*

$$ \mathcal{D}^{\varkappa}_{l^a}\sigma = \sum_i \Big[\frac{\partial\sigma^i(\varkappa(l))}{\partial l^a} + \sum_j {}_a\Gamma^i{}_j(l;\varkappa)\sigma^j(\varkappa(l))\Big] e_i(l), \tag{A.11}$$

where $\{e_i(l)\}$ is a basis in $\pi^{-1}(\varkappa(l))$, $\sigma(\varkappa(l)) =: \sum_i \sigma^i(\varkappa(l))e_i(l)$, and the ***coefficients*** *of L are defined by*

$$ {}_a\Gamma^i{}_j(l;\varkappa) := \left.\frac{\partial L^i{}_j(l,m;\varkappa)}{\partial m^a}\right|_{m=l} = -\left.\frac{\partial L^i{}_j(m,l;\varkappa)}{\partial m^a}\right|_{m=l}. \tag{A.12}$$

Here $L^j{}_i(\cdots)$ are the components of L, $L^{\varkappa}_{l\to m}e_i(l) =: \sum_j L^j{}_i(m,l;\varkappa)e_j(m)$.

The above general definitions and results will be used in this work in the special case of linear transports along the identity map of the bundle's base.

[14]We present below, in definition A.2, directly the definition of a *section*-derivation along injective mapping $\varkappa$ as only it will be employed in the present paper (for $\varkappa = \mathsf{id}_N$). If $\varkappa$ is not injective, the mapping (A.7) could be multiple-valued at the points of self-intersection of $\varkappa$, if any. For some details when $\varkappa$ is an arbitrary path in N, see [**?**, subsect. 3.3].

Appendix B. Matrix operators

In this appendix we point to some peculiarities of linear (matrix) operators acting on $n\times 1$, $n\in\mathbb{N}$, matrix fields over the space-time M. Such operators naturally appear in the theory of Dirac equation where one often meets 4×4 matrices whose elements are operators; e.g. an operator of this kind is $\not\partial := \gamma^\mu\partial_\mu = \big[(\gamma^\mu)^\alpha_{\ \beta}\partial_\mu\big]^3_{\alpha,\beta=0}$ where γ^μ are the well known Dirac γ-matrices [?, ?].

We call an $n\times n$, $n\in\mathbb{N}$ matrix $B=[b^\alpha_{\ \beta}]^n_{\alpha,\beta=1}$ a *(linear) matrix operator*[15] if $b^\alpha_{\ \beta}$ are (linear) operators acting on the space K^1 of C^1 functions $f\colon M\to\mathbb{C}$. If $\{f^0_\nu\}$ is a basis in the set $M(n,1)$ of $n\times 1$ matrices with the μ-th element of f^0_ν being $(f^0_\nu)^\mu := \delta^\mu_\nu$,[16] then by definition

$$B\psi := B(\psi) := B\cdot(\psi) := \sum_{\alpha,\beta=1}^{n}\big(b^\alpha_{\ \beta}(\psi^\beta_0\big)f^0_\alpha, \qquad \psi=\psi^\beta_0 f^0_\beta\in M(n,1). \tag{B.1}$$

For instance, we have $\not\partial\psi = \sum^3_{\alpha,\beta,\mu=0}(\gamma^\mu)^\alpha_{\ \beta}(\partial_\mu\psi^\beta_0)f^0_\alpha$.

To any constant matrix $C=[c^\alpha_{\ \beta}]$, $c^\alpha_{\ \beta}\in\mathbb{C}$, corresponds a matrix operator $\overline{C} := [c^\alpha_{\ \beta}\,\mathsf{id}_{K^1}]$. Since $C\psi\equiv\overline{C}\psi$ for any ψ, we identify C and $\overline{C}$ and will make no difference between them.

The multiplication of matrix operators, denoted by $\odot$, is a combination of matrix multiplication (denoted by $\cdot$) and maps (operators) composition (denoted by $\circ$). If $A=[a^\alpha_{\ \beta}]$ and $B=[b^\alpha_{\ \beta}]$ are matrix operators, the product of A and B is also a matrix operator such that

$$AB := A\odot B := \Big[\sum_\mu a^\alpha_{\ \mu}\circ b^\mu_{\ \beta}\Big]. \tag{B.2}$$

One can easily show that this is an associative operation. It is linear in the first argument and, if its first argument is linear matrix operator, then it is linear in its second argument too. For constant matrices (see above) the multiplication (B.2) coincides with the usual matrix one.

Let $\{f_\alpha(x)\}$ be a basis in $M(n,1)$ depending on $x\in M$ and $f(x) := [f^\beta_\alpha(x)]$ be defined by the expansion $f_\alpha(x)=f^\beta_\alpha(x)f^0_\beta$. The *matrix of a matrix operator* $B=[b^\alpha_{\ \beta}]$ with respect to $\{f_\alpha\}$, i.e. the matrix of the matrix elements of B considered as an operator, is also a matrix operator $\boldsymbol{B} := [B^\alpha_{\ \beta}]$ such that

$$B\psi|_x =: \sum_{\alpha,\beta}\big(B^\alpha_{\ \beta}(\psi^\beta)\big)\big|_x f_\alpha(x), \qquad \psi(x)=\psi^\beta(x)f_\beta(x). \tag{B.3}$$

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[15] Also a good term for such an object is (linear) *matrixor*.

[16] $\delta^\mu_\nu = 0$ for $\mu\neq\nu$ and $\delta^\mu_\nu=1$ for $\mu=\nu$. From here to equation (B.10) in this appendix the Greek indices run from 1 to $n\geq 1$.

Therefore

$$\varphi = B\psi \iff \boldsymbol{\varphi} = \boldsymbol{B}\boldsymbol{\psi}, \tag{B.4}$$

where $\boldsymbol{\psi} \in M(n, 1)$ is the matrix of the components of ψ in the basis given. Comparing (B.1) and (B.3), we get

$$B^{\alpha}{}_{\beta} = \sum_{\mu,\nu} \big(f^{-1}(x)\big)^{\alpha}_{\mu} b^{\mu}{}_{\nu} \circ \big(f^{\nu}_{\beta}(x)\,\mathsf{id}_{K^1}\big). \tag{B.5}$$

For any matrix $C = [c^{\alpha}{}_{\beta}] \in GL(n, \mathbb{C})$, considered as a matrix operator (see above), we have

$$\boldsymbol{C} = \big[C^{\alpha}{}_{\beta}(x)\big], \quad C^{\alpha}{}_{\beta}(x) = \big(f^{-1}(x)\big)^{\alpha}_{\mu} c^{\mu}{}_{\nu} f^{\nu}_{\beta}, \quad C(f_{\beta}) = C^{\alpha}{}_{\beta} f_{\alpha}. \tag{B.6}$$

Therefore, as one can expect, the matrix of the unit matrix $\mathbb{1}_n = \big[\delta^{\beta}_{\alpha}\big]^{n}_{\alpha,\beta=1}$ is exactly the unit matrix:

$$\boldsymbol{\mathbb{1}}_n = \mathbb{1}_n. \tag{B.7}$$

If $\{f_{\mu}(x)\}$ does not depend on x, i.e. if f^{β}_{α} are complex constants, and $b^{\mu}{}_{\nu}$ are linear, than (B.5) implies

$$B^{\alpha}{}_{\beta} = \big(f^{-1}(x)\big)^{\alpha}_{\mu} f^{\nu}_{\beta}(x) b^{\mu}{}_{\nu} \quad \text{for} \quad \partial f^{\beta}_{\alpha}(x)/\partial x^{\mu} = 0. \tag{B.8}$$

In particular, for a linear matrix operator B, we have

$$\boldsymbol{B} = B \quad \text{for} \quad f^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} \quad \text{(i.e. for } f_{\alpha}(x) = f^{0}_{\alpha}\text{)}. \tag{B.9}$$

Combining (B.2) and (B.5), we deduce that the matrix of a product of matrix operators is the product of the corresponding matrices:

$$C = A \odot B \quad \iff \quad \boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} = \boldsymbol{A} \odot \boldsymbol{B}. \tag{B.10}$$

After a simple calculation, we find the matrices of $\mathbb{1}_4\partial_{\mu}$ and $\partial\!\!\!/ = \gamma^{\mu}\partial_{\mu}$:[17]

$$\boldsymbol{\partial}_{\mu} = \mathbb{1}_4\partial_{\mu} + \mathcal{E}_{\mu}(x), \qquad \boldsymbol{\partial\!\!\!/} = \mathcal{G}^{\mu}(x)\big(\mathbb{1}_4\partial_{\mu} + \mathcal{E}_{\mu}(x)\big) = \mathcal{G}^{\mu}(x)\boldsymbol{\partial}_{\mu} \tag{B.11}$$

where $\mathcal{G}^{\mu}(x) = \big[\mathcal{G}^{\lambda\mu}_{\ \ \alpha}(x)\big]^{3}_{\alpha,\lambda=0}$ and $\mathcal{E}_{\mu}(x) = \big[\mathcal{E}^{\lambda}_{\mu\alpha}(x)\big]^{3}_{\alpha,\lambda=0}$ are defined via the expansions

$$\gamma^{\mu} f_{\nu}(x) =: \mathcal{G}^{\lambda\mu}_{\ \ \nu}(x) f_{\lambda}(x), \quad \partial_{\mu} f_{\nu}(x) =: \mathcal{E}^{\lambda}_{\mu\nu}(x) f_{\lambda}(x). \tag{B.12}$$

So, $\mathcal{G}^{\mu}$ is the matrix of γ^{μ} considered as a matrix operator (see (B.6)).[18] Evidently

$$\boldsymbol{\partial\!\!\!/} = \partial\!\!\!/ \quad \text{for} \quad f_{\alpha}(x) = f^{0}_{\alpha}. \tag{B.13}$$

[17]Here and below the Greek indices run from 0 to 3.

[18]We denote the matrix of γ^{μ} by $\mathcal{G}^{\mu}$ instead of $\boldsymbol{\gamma}^{\mu}(x)$ as usually [?, ?] by $\boldsymbol{\gamma}$ is denoted the matrix 3-vector $(\gamma^1, \gamma^2, \gamma^3)$.

Combining (B.6),(B.7), and (B.11), we get that the matrix of the matrix operator $\mathrm{i}\hbar \not{D} - mc\mathbb{1}_4$, entering in the Dirac equation (2.1), is

$$\mathrm{i}\hbar \not{\boldsymbol{D}} - mc\boldsymbol{1}_4 = \mathrm{i}\hbar \mathcal{G}^\mu(x)\big(\mathbb{1}_4 D_\mu + \mathcal{E}_\mu(x)\big) - mc\mathbb{1}_4, \tag{B.14}$$

so that

$$\mathrm{i}\hbar \not{\boldsymbol{D}} - mc\boldsymbol{1}_4 = \mathrm{i}\hbar \not{D} - mc\mathbb{1}_4 \quad \text{for} \quad f_\alpha(x) = f_\alpha^0. \tag{B.15}$$